汎用 3 維邊界元素法

1　積分方程式表示

座標原點 0 定於水平面(xy)，z 軸垂直向上的封閉領域內，流體假定為非壓縮性非粘性的理想流體，流體運動為非迴轉性的無渦運動，流體從靜止狀態開始運動，即造波水槽的流體運動具有滿足下列 Laplace 方程式的速度勢ϕ(x，y，z；t)

$$\frac{\partial \phi^2}{\partial x^2}+\frac{\partial \phi^2}{\partial y^2}+\frac{\partial \phi^2}{\partial z^2}=0 \qquad (1)$$

根據 Green 定理，流體內任意 1 點的速度勢ϕ，可由邊界上的速度勢及其在法線方向的導函數以下式積分方程式表示

$$\gamma\,\phi+\int \phi \bar{q}^{*} dA=\int \bar{\phi} q^{*} dA \qquad (2)$$

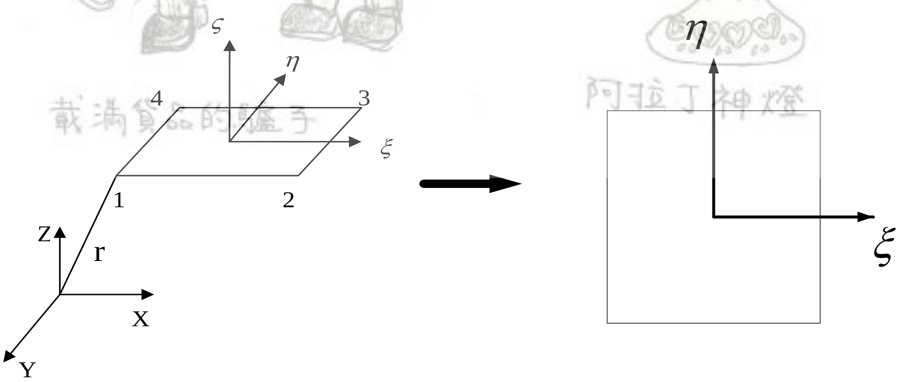
但

$$q^{*}=\frac{1}{4\pi r} \quad ， \quad \bar{q}^{*}=\frac{\partial q^{*}}{\partial n} \qquad (3)$$

r 為流體內任意 1 點與邊界上任意 1 點間的距離，n 表示向外法線。對內任意 1 點領域 γ=1，對平滑的邊界，當任意一點接近邊界上時，γ=0.5。

2　積分方程式離散化

由於(2)式無法求得解析解必須利用數值分析，本文將採用平面一次元素進行離散化。



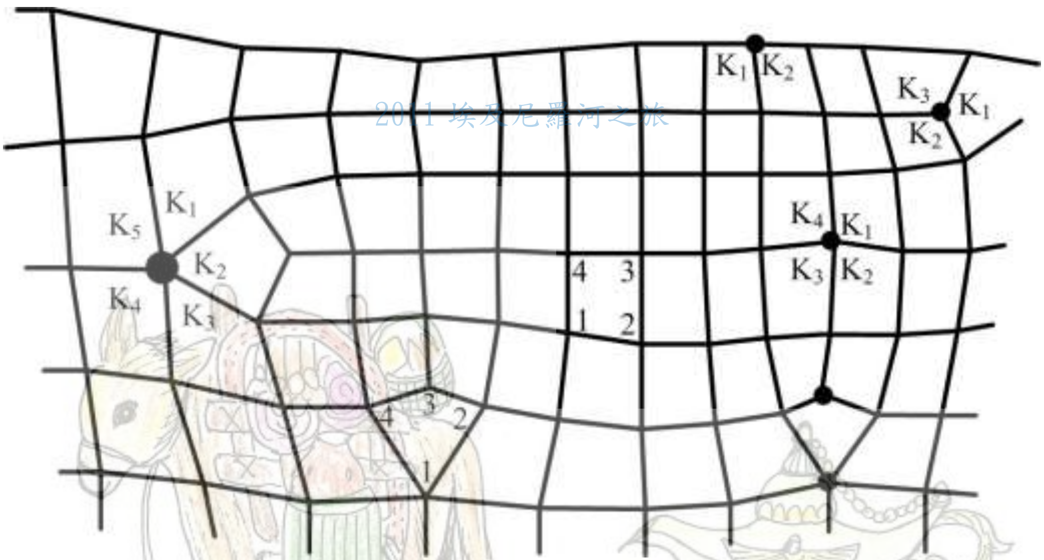
如上圖，對某一特定元素內任意 1 點的 φ 可以下式表示

$$\phi=\sum_{i=1}^{4}\psi_i\phi_i \tag{4}$$

但 $\psi_i(i=1\sim4)$ 為下式所示形狀函數。

$$\left.\begin{array}{l}\psi_1=\frac{1}{4}(1-\xi)(1-\eta),\psi_2=\frac{1}{4}(1+\xi)(1-\eta)\\ \\ \psi_3=\frac{1}{4}(1+\xi)(1+\eta),\psi_4=\frac{1}{4}(1-\xi)(1+\eta)\end{array}\right\} \tag{5}$$

為求解邊界上的速度勢及其法線方向導函數值，首先必須對(3)式，在 γ=0.5，即源點(source point)在邊界上時進行數值分析。



若對邊界面 n 個元素(nt 節點)組成的平面 1 次元素加以離散，對被積分節點 j 而言，其相鄰元素如上圖，以 $k_1\sim k_{ns}$ 表示，將(4)式代入(3)式得下列和分方程式。

$$\phi_i+\sum_{j=1}^{n_t}\sum_{s=1}^{n_s}h_{ij}{}^s\phi_j=\sum_{j=1}^{nt}\sum_{s=1}^{n_s}g_{ij}{}^s\overline{\phi}_j \tag{6}$$

式中

$$h_{ij}^{\ s} = \frac{1}{\gamma}\int_{\Gamma_{ks}} \psi_s \bar{q}^* dA$$

$$= -\frac{1}{8\pi}\int_{-1}^{1}\int_{-1}^{1}\psi_s \frac{1}{r^2}\frac{\partial r}{\partial n}|G|_{\Gamma_{ks}}\, d\xi d\eta \qquad (\mathrm{s}=1\sim \mathrm{n_s}) \qquad (7)$$

$$g_{ij}^{\ s} = \frac{1}{\gamma}\int_{\Gamma_{ks}} \psi_s q^* dA$$

$$= \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\psi_s q^* |G|_{\Gamma_{ks}}\, d\xi d\eta \qquad (\mathrm{s}=1\sim \mathrm{n_s}) \qquad (8)$$

但

$$r = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} \qquad (9)$$

$$\frac{\partial r}{\partial n} = \frac{\partial r}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial n} + \frac{\partial r}{\partial z}\frac{\partial z}{\partial n} \qquad (10)$$

對各被積分元素

$$|G|_{\Gamma_{ks}} = \sqrt{g_1^{\ 2} + g_2^{\ 2} + g_3^{\ 2}} \qquad (\mathrm{s}=1\sim 4) \qquad (11)$$

$$\left.\begin{aligned} g_1 &= \frac{\partial y}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \xi}\frac{\partial y}{\partial \eta} \\ g_2 &= \frac{\partial z}{\partial \xi}\frac{\partial x}{\partial \eta} - \frac{\partial x}{\partial \xi}\frac{\partial z}{\partial \eta} \\ g_3 &= \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} \end{aligned}\right\} \qquad (12)$$

當 i ≠ j 時，每個元素的 4 個節點以逆時針方向用 1～4 表示並採用 Guass 積分進行數值積分時，當被積分點 j 與第 s 相鄰元素的第 n 節點(n=1～4)重疊，

n=1 時

$$h_{ij}^{\ s} = -\frac{1}{8\pi}\sum_{l=1}^{n}\sum_{m=1}^{n} w_l w_m (1-\xi_l-\eta_m+\xi_l\eta_m)\frac{1}{r_{ilm}^{\ 2}}\frac{\partial r_{ilm}}{\partial n}|G|_{\Gamma_{k1}} \qquad (13)$$

$$g_{ij}^{\ s}=\frac{1}{8\pi}\sum_{l=1}^{n}\sum_{m=1}^{n}w_l w_m\left(1-\xi_l-\eta_m+\xi_l\eta_m\right)\frac{1}{r_{ilm}}\left|G\right|_{\Gamma_{k1}} \qquad (14)$$

n=2 時

$$h_{ij}^{\ s}=-\frac{1}{8\pi}\sum_{l=1}^{n}\sum_{m=1}^{n}w_l w_m\left(1+\xi_l-\eta_m-\xi_l\eta_m\right)\frac{1}{r_{ilm}^{\ 2}}\frac{\partial r_{ilm}}{\partial n}\left|G\right|_{\Gamma_{k2}} \qquad (15)$$

$$g_{ij}^{\ s}=\frac{1}{8\pi}\sum_{l=1}^{n}\sum_{m=1}^{n}w_l w_m\left(1+\xi_l-\eta_m-\xi_l\eta_m\right)\frac{1}{r_{ilm}}\left|G\right|_{\Gamma_{k2}} \qquad (16)$$

n=3 時

$$h_{ij}^{\ s}=-\frac{1}{8\pi}\sum_{l=1}^{n}\sum_{m=1}^{n}w_l w_m\left(1+\xi_l+\eta_m+\xi_l\eta_m\right)\frac{1}{r_{ilm}^{\ 2}}\frac{\partial r_{ilm}}{\partial n}\left|G\right|_{\Gamma_{k3}} \qquad (17)$$

$$g_{ij}^{\ s}=\frac{1}{8\pi}\sum_{l=1}^{n}\sum_{m=1}^{n}w_l w_m\left(1+\xi_l+\eta_m+\xi_l\eta_m\right)\frac{1}{r_{ilm}}\left|G\right|_{\Gamma_{k3}} \qquad (18)$$

n=4 時

$$h_{ij}^{\ s}=-\frac{1}{8\pi}\sum_{l=1}^{n}\sum_{m=1}^{n}w_l w_m\left(1-\xi_l+\eta_m-\xi_l\eta_m\right)\frac{1}{r_{ilm}^{\ 2}}\frac{\partial r_{ilm}}{\partial n}\left|G\right|_{\Gamma_{k4}} \qquad (19)$$

$$g_{ij}^{\ s}=\frac{1}{8\pi}\sum_{l=1}^{n}\sum_{m=1}^{n}w_l w_m\left(1-\xi_l+\eta_m-\xi_l\eta_m\right)\frac{1}{r_{ilm}}\left|G\right|_{\Gamma_{k4}} \qquad (20)$$

式中

$$\frac{\partial r_{ilm}}{\partial n}=\frac{x_{lm}-x_i}{r_{ilm}}\left(\frac{\partial x}{\partial n}\right)_j+\frac{y_{lm}-y_i}{r_{ilm}}\left(\frac{\partial y}{\partial n}\right)_j+\frac{z_{lm}-z_i}{r_{ilm}}\left(\frac{\partial z}{\partial n}\right)_j$$

式中 r_{ilm} 為源點 i 至被積分元素(j)的 Guass 積分點(ξ_l，η_m)間的距離，w_l、w_m 為加權函數，n=2 時，$w_l=w_m=1$。

當 i = j 時，由於$\partial r/\partial n$=0 得

$$h_{ij}^{\ s}=0 \qquad (s=1\sim 4) \qquad (21)$$

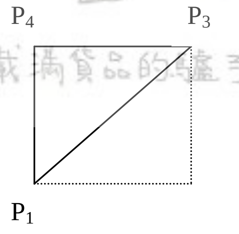


圖 4a

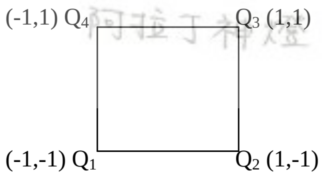


圖 4b

對(17)～(20)式，當 i = j 時會產生特異值，必須作下列處理，如圖 4a 所示，對某一被積分元素，其節點為P_1、P_2、P_3及，P_4，若討論節點為p_1時，先將四邊形元素分割成三角形元素 $\Delta P_1 P_3 P_4$，再將其保角變換成如圖 4b 所示正方形元素，2 者間的座標關係如下。

$$x=\sum_{k=1}^{4}\phi^k\tilde{x}^k \tag{22}$$

$\tilde{x}^k$ (k=1～4)為Q_1～Q_4點在實際 3 度空間內的座標，而

$$\left.\begin{aligned}\phi^1&=\frac{1}{4}(1-\xi)(1-\eta),\phi^2=\frac{1}{4}(1+\xi)(1-\eta)\\\phi^3&=\frac{1}{4}(1+\xi)(1+\eta),\phi^4=\frac{1}{4}(1-\xi)(1+\eta)\end{aligned}\right\} \tag{23}$$

P_k點的座標為x^k時

$$\left.\begin{aligned}x^1&=\tilde{x}^1=\tilde{x}^2\\x^3&=\tilde{x}^3\\x^4&=\tilde{x}^4\end{aligned}\right\} \tag{24}$$

將(24)式代入(23)式得

$$\begin{aligned}x&=\phi^1x^1+\phi^2x^2+\phi^3x^3+\phi^4x^4\\&=\frac{1}{2}(1-\eta)x^1+\phi^3x^3+\phi^4x^4\\&=\sum_{k=1}^{3}\psi^kx^k\end{aligned} \tag{25}$$

式中

$$\psi^1=\frac{1}{2}(1-\eta)$$

$$\psi^2=\phi^3=\frac{1}{4}(1+\xi)\ (1+\eta)$$

$$\psi^3=\phi^4=\frac{1}{4}(1-\xi)(1+\eta)$$

因源點座標為x^1，故源點元素內任意 1 點的位置向量 r 為

$$r = x - x^1$$

$$= \left(\sum_{k=1}^{3} \psi^k x^k \right) - x^1$$

$$= \frac{1}{2}(1+\eta)\left(\phi_*^1 x^1 + \phi_*^3 x^3 + \phi_*^4 x^4\right)$$

$$= \rho \times r^* \qquad (26)$$

式中

$$\rho = \frac{(1+\eta)}{2} \qquad (27)$$

$$r^* = \phi_*^1 x^1 + \phi_*^3 x^3 + \phi_*^4 x^4 \qquad (28)$$

但

$$\left.\begin{aligned} \phi_*^1 &= -1 \\ \phi_*^3 &= \frac{1}{2}(1+\xi) \\ \phi_*^4 &= \frac{1}{2}(1-\xi) \end{aligned}\right\} \qquad (29)$$

故得源點與元素內任意 1 點間的距離 r

$$r = |\rho| r^* \qquad (30)$$

但 $r^* = \sqrt{r_1^{*2} + r_2^{*2} + r_3^{*2}}$，$r_1^*$、$r_2^*$ 及 r_3^* 各為 r^* 在 x、y 及 z 方向之分量。

當 $x \to x^1$ 時，$|\rho| \to 0$ 但 $r^* \neq 0$，因此對 ξ，η 座標的平面元素 $d\Gamma$，得

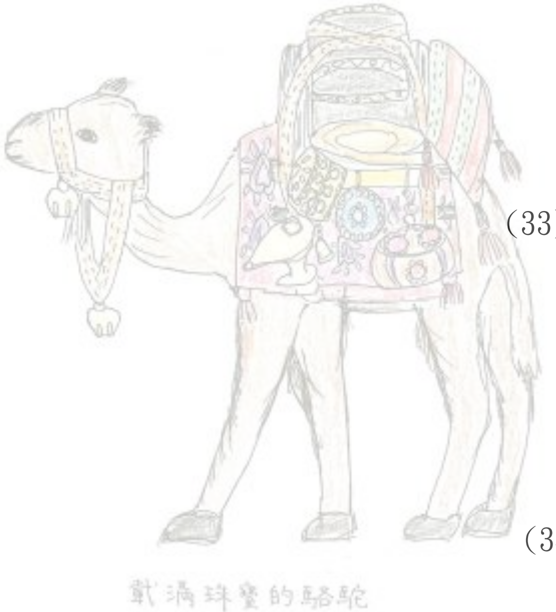
$$d\Gamma = \left| \frac{\partial \vec{r}}{\partial \xi} \times \frac{\partial \vec{r}}{\partial \eta} \right| d\xi d\eta$$

$$= |\rho| \left| \left(\sum_{k=3}^{4} \frac{\partial \phi_*^k}{\partial \xi} x^k \right) \times \frac{\partial \vec{r}}{\partial \eta} \right| d\xi d\eta$$

$$= |\rho| \left| \frac{\partial \vec{r}^*}{\partial \xi} \times \frac{\partial \vec{r}}{\partial \eta} \right| d\xi d\eta \qquad (31)$$

即得

$$\frac{1}{r}d\Gamma = \frac{1}{r^*}\left|\frac{\partial \vec{r}^*}{\partial \xi} \times \frac{\partial \vec{r}}{\partial \eta}\right| d\xi d\eta \tag{32}$$



$$\left.\begin{aligned} \frac{\partial x^*}{\partial \xi} &= \frac{1}{2}\left(x^3 - x^4\right) \\ \frac{\partial y^*}{\partial \xi} &= \frac{1}{2}\left(y^3 - y^4\right) \\ \frac{\partial z^*}{\partial \xi} &= \frac{1}{2}\left(z^3 - z^4\right) \end{aligned}\right\} \tag{33}$$

由上式可知特異性已被消除。

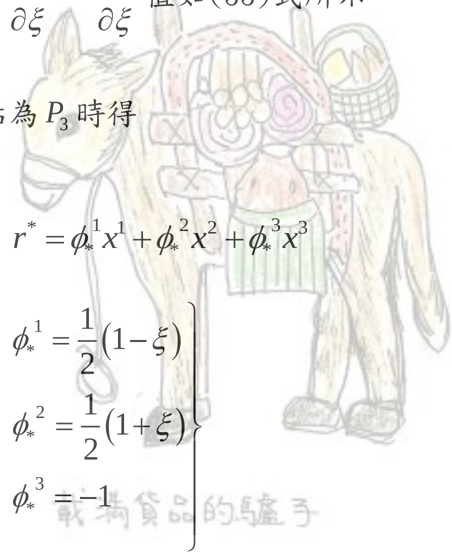
同理，當源點為 P_2 時得

$$r^* = \phi_*^2 x^2 + \phi_*^3 x^3 + \phi_*^4 x^4 \tag{34}$$

$$\left.\begin{aligned} \phi_*^2 &= -1 \\ \phi_*^3 &= \frac{1}{2}\left(1+\xi\right) \\ \phi_*^4 &= \frac{1}{2}\left(1-\xi\right) \end{aligned}\right\} \tag{35}$$

$\frac{\partial x^*}{\partial \xi}$、$\frac{\partial y^*}{\partial \xi}$、$\frac{\partial z^*}{\partial \xi}$ 值如(33)式所示。

當源點為 P_3 時得



$$r^* = \phi_*^1 x^1 + \phi_*^2 x^2 + \phi_*^3 x^3 \tag{36}$$

$$\left.\begin{aligned} \phi_*^1 &= \frac{1}{2}\left(1-\xi\right) \\ \phi_*^2 &= \frac{1}{2}\left(1+\xi\right) \\ \phi_*^3 &= -1 \end{aligned}\right\} \tag{37}$$

$$\left.\begin{aligned}\frac{\partial x^*}{\partial \xi} &= \frac{1}{2}\left(x^2 - x^1\right)\\ \frac{\partial y^*}{\partial \xi} &= \frac{1}{2}\left(y^2 - y^1\right)\\ \frac{\partial z^*}{\partial \xi} &= \frac{1}{2}\left(z^2 - z^1\right)\end{aligned}\right\} \tag{38}$$

當源點為 P_4 時得

$$r^* = \phi_*^1 x^1 + \phi_*^2 x^2 + \phi_*^4 x^4 \tag{39}$$

$$\left.\begin{aligned}\phi_*^1 &= \frac{1}{2}(1-\xi)\\ \phi_*^2 &= \frac{1}{2}(1+\xi)\\ \phi_*^4 &= -1\end{aligned}\right\}$$

$\frac{\partial x^*}{\partial \xi}$、$\frac{\partial y^*}{\partial \xi}$、$\frac{\partial z^*}{\partial \xi}$ 值如 38)式所示。

當被積分點 j 與第 s 相鄰元素的第 n 節點(n=1～4)重疊，

n=1 時

$$g_{ij}^{\ s} = \frac{1}{8\pi}\sum_{l=1}^{n}\sum_{m=1}^{n} w_l w_m \left(1-\xi_l-\eta_m+\xi_l\eta_m\right)\frac{1}{r^*_{ilm}}\left|G^*\right|_{\Gamma_{k1}} \tag{40}$$

n=2 時

$$g_{ij}^{\ s} = \frac{1}{8\pi}\sum_{l=1}^{n}\sum_{m=1}^{n} w_l w_m \left(1+\xi_l-\eta_m-\xi_l\eta_m\right)\frac{1}{r^*_{ilm}}\left|G^*\right|_{\Gamma_{k2}} \tag{41}$$

n=3 時

$$g_{ij}^{\ s} = \frac{1}{8\pi}\sum_{l=1}^{n}\sum_{m=1}^{n} w_l w_m \left(1+\xi_l+\eta_m+\xi_l\eta_m\right)\frac{1}{r^*_{ilm}}\left|G^*\right|_{\Gamma_{k3}} \tag{42}$$

n=4 時

$$g_{ij}^{\ s} = \frac{1}{8\pi}\sum_{l=1}^{n}\sum_{m=1}^{n} w_l w_m \left(1-\xi_l+\eta_m-\xi_l\eta_m\right)\frac{1}{r^*_{ilm}}\left|G^*\right|_{\Gamma_{k4}} \tag{43}$$

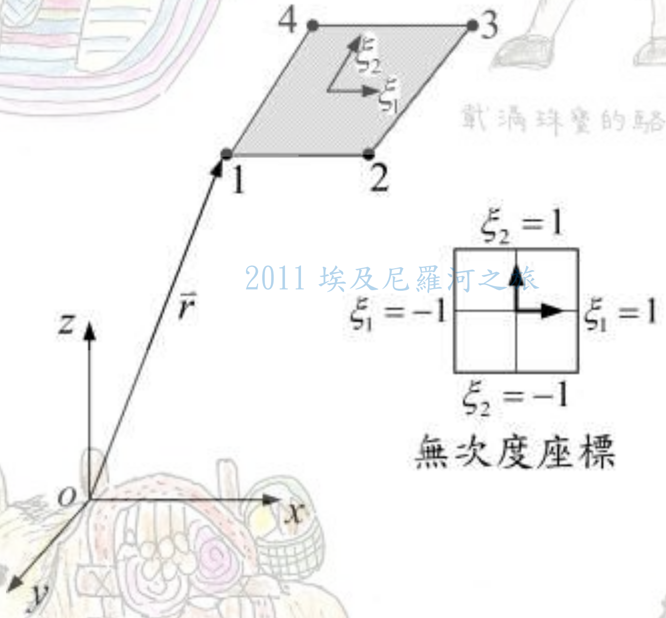
式中

$$\left|G^*\right|_{\Gamma_{ks}} = \sqrt{g_1^{*2}+g_2^{*2}+g_3^{*2}} \qquad (s=1\sim4) \tag{44}$$

$$\left.\begin{aligned} g_1^* &= \frac{\partial y^*}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial z^*}{\partial \xi}\frac{\partial y}{\partial \eta} \\ g_2^* &= \frac{\partial z^*}{\partial \xi}\frac{\partial x}{\partial \eta} - \frac{\partial x^*}{\partial \xi}\frac{\partial z}{\partial \eta} \\ g_3^* &= \frac{\partial x^*}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y^*}{\partial \xi}\frac{\partial x}{\partial \eta} \end{aligned}\right\} \tag{45}$$

式中 r^*_{ilm} 為源點 i 至被積分元素(j)的 Guass 積分點(ξ_l，η_m)間的距離。

3　全體座標與元素座標間的關係



如上圖所示的座標系，函數 φ 的變換法則為

$$\begin{bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \phi}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \dfrac{\partial \phi}{\partial \xi} \\ \dfrac{\partial \phi}{\partial \eta} \\ \dfrac{\partial \phi}{\partial \zeta} \end{bmatrix} \tag{46}$$

$$J=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta}\\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}\end{bmatrix} \tag{47}$$

又

$$\left.\begin{aligned}\frac{\partial x}{\partial \xi}&=\frac{1}{4}\left[-(1-\eta)x_1+(1-\eta)x_2+(1+\eta)x_3-(1+\eta)x_4\right]\\ \frac{\partial x}{\partial \eta}&=\frac{1}{4}\left[-(1-\xi)x_1-(1+\xi)x_2+(1+\xi)x_3+(1-\xi)x_4\right]\end{aligned}\right\} \tag{48}$$

$$\left.\begin{aligned}\frac{\partial y}{\partial \xi}&=\frac{1}{4}\left[-(1-\eta)y_1+(1-\eta)y_2+(1+\eta)y_3-(1+\eta)y_4\right]\\ \frac{\partial y}{\partial \eta}&=\frac{1}{4}\left[-(1-\xi)y_1-(1+\xi)y_2+(1+\xi)y_3+(1-\xi)y_4\right]\end{aligned}\right\} \tag{49}$$

$$\left.\begin{aligned}\frac{\partial z}{\partial \xi}&=\frac{1}{4}\left[-(1-\eta)z_1+(1-\eta)z_2+(1+\eta)z_3-(1+\eta)z_4\right]\\ \frac{\partial z}{\partial \eta}&=\frac{1}{4}\left[-(1-\xi)z_1-(1+\xi)z_2+(1+\xi)z_3+(1-\xi)z_4\right]\end{aligned}\right\} \tag{50}$$

4　連立方程式

將(6)式以下列矩陣形式表示

$$[\phi]=[O][\bar{\phi}] \tag{51}$$

但

$$[O]=[H+I]^{-1}[G]$$

$$[H]=\sum_{s=1}^{ns} h_{ij}^{\ s}$$

$$[G]=\sum_{s=1}^{ns} g_{ij}^{\ s}$$

上式為利用一次元素的積分方程式離散形式