非定常移流擴散使用含時間項的基本解

非定常移流擴散的控制方程式如下

$$\dot{C}+v_jC_{,j}-kC_{,jj}=0$$

C=濃度，k=擴散係數，v_j=流速分量。將上式無因次化得

$$Pe\left(\dot{\theta}+u_j\theta_{,j}\right)-\theta_{,jj}=0 \qquad (1)$$

$Pe=LU/k$，L=代表長，U=基準流速

$$u_j=v_j/U$$

$$\theta=\left(C-C_{\min}\right)/\left(C_{Max}-C_{\min}\right)$$

將(1)式乘以含時間項的基本解 w=w(r, τ)，並對空間及時間作積分，得

$$\int_{\tau_1}^{\tau_2}\int_{\Omega}w\left[Pe\left(\dot{\theta}+u_j\theta_{,j}\right)-\theta_{,jj}\right]d\Omega d\tau$$

對上式中的時間微分項及擴散項作部份積分得

$$\begin{aligned}&\int_{\tau_1}^{\tau_2}\int_{\Omega}w\left[Pe\left(\dot{\theta}+u_j\theta_{,j}\right)-\theta_{,jj}\right]d\Omega d\tau=\\&\int_{\tau_1}^{\tau_2}\int_{\Omega}w\left[\frac{\partial(wPe\theta)}{\partial\tau}-\frac{\partial(wPe)}{\partial\tau}\theta\right]d\Omega d\tau+\int_{\tau_1}^{\tau_2}\int_{\Omega}wPeu_j\theta_{,j}d\Omega d\tau\\&-\int_{\tau_1}^{\tau_2}\int_{\Omega}\left[\left(w\theta_{,j}\right)_{,j}-\left(w_{,j}\theta\right)_{,j}+w_{,jj}\theta\right]d\Omega d\tau=0\end{aligned}$$

因

$$\begin{aligned}&\int_{\tau_1}^{\tau_2}\int_{\Omega}w\left[Pe\left(\dot{\theta}+u_j\theta_{,j}\right)-\omega_{,jj}\right]d\Omega d\tau\\&=\int_{\Omega}\left[wPe\theta\right]_{\tau_1}^{\tau_2}d\Omega+\int_{\tau_1}^{\tau_2}\int_{\Omega}wPeu_j\theta_{,j}d\Omega d\tau\\&-\int_{\tau_1}^{\tau_2}\int_{\Gamma}\left(w\theta_{,n}-w_{,n}\theta\right)d\Gamma d\tau\\&-\int_{\tau_1}^{\tau_2}\int_{\Omega}\left[\frac{\partial(wPe)}{\partial\tau}+w_{,jj}\right]\theta d\Omega d\tau=0\end{aligned} \qquad (2)$$

當加權函數採用下式所示函數時

$$w = \frac{Pe}{4\pi\left(\tau_2 - \tau\right)} \exp\left[-\frac{Per^2}{4\left(\tau_2 - \tau\right)}\right] \qquad (3)$$

(2)式中的第 1 項及第 4 項可作下列變形

$$\begin{aligned} &\int_\Omega \left[wPe\theta\right]_{\tau_1}^{\tau_2} d\Omega - \int_{\tau_1}^{\tau_2}\int_\Omega \left[\frac{\partial\left(wPe\right)}{\partial\tau} + w_{,jj}\right]\theta d\Omega d\tau = \\ &-\int_\Omega \left(wPe\theta\right)_{\tau=\tau_1} d\Omega + \theta\left(P,\tau_2\right)Pe \end{aligned} \qquad (4)$$

上式右邊第 2 項的 P 點為考量點。

當加權函數採用(3)式時，可由下列推導求得(4)式。首先將(4)式中的[ ]內的式子以極座標表示，權重 w 為至原點 P 間的距離 r 的函數時，因

$$\frac{\partial\left(wPe\right)}{\partial\tau} + w_{,jj} = \frac{\partial\left(wPe\right)}{\partial\tau} + \frac{\partial w}{r\partial r} + \frac{\partial^2 w}{\partial r^2}$$

將(3)式代入上式可得

$$\frac{\partial\left(wPe\right)}{\partial\tau} + w_{,jj} = 0$$

當極座標原點 P 接近積分點 Q(χ_1, χ_2)，同時時間 τ 亦接近 τ_2 時，(4)式左邊(P, τ_2)近傍的濃度 $\theta\left(r,\tau\right)$ 以 $\theta\left(0,\tau_2\right)$ 取代，則對其近傍微小領域 dΩ 可作下列展開

$$\int\left[wPe\theta\right]_{\tau_1}^{\tau_2} d\Omega - \int_{\tau_1}^{\tau_2}\int\left[\frac{\partial\left(wPe\right)}{\partial\tau} + \theta_{,jj}\right]\theta d\Omega d\tau$$

$$= \int\left[wPe\theta\right]_{\tau_1}^{\tau_2} rdrd\alpha - \theta\left(0,\tau_2\right)\int_{\tau_1}^{\tau_2}\int\frac{\partial\left(wPe\right)}{\partial\tau} rdrd\alpha d\tau - \theta\left(0,\tau_2\right)\int_{\tau_1}^{\tau_2}\int w_{,jj} rdrd\alpha d\tau$$

$$= \int\left[wPe\theta\right]_{\tau_1}^{\tau_2} rdrd\alpha - \theta\left(0,\tau_2\right)\int_{\tau_1}^{\tau_2}\left[wPe\right]_{\tau_1}^{\tau_2} rdrd\alpha - \theta\left(0,\tau_2\right)\int_{\tau_1}^{\tau_2}\left[r\frac{\partial w}{\partial r}\right]_0^r r2\pi dr$$

$$= -\int\left(wPe\theta\right)_{\tau_1} rdrd\alpha + \omega\left(0,\tau_2\right)\int\left(wPe\theta\right)_{\tau_1} rdrd\alpha + \theta\left(0,\tau_2\right)\int_{\tau_1}^{\tau_2}\frac{\partial w}{\partial r}2\pi rd\tau$$

上式右邊第 1 項及第 2 項當 r→0 時為零，對第 3 項作時間積分後，令 r→0 時，得$\theta(0,\tau_2)$Pe。當極座標原點 P 不接近積分點 Q(χ_1,χ_2)，即 r 不為 0，而 $\tau\to 0$ 時(4)式左邊第 1 項為 0，故(3)式可改寫成下列形式

$$\theta(r,\tau_2)Pe+\int_{\tau_1}^{\tau_2}\int_\Gamma w_{,n}\theta\, d\Gamma d\tau=\int_{\tau_1}^{\tau_2}\int_\Gamma \theta_{,n} w\, d\Gamma d\tau-\int_{\tau_1}^{\tau_2}\int_\Omega wPe\,u_j\theta_{,j}\, d\Omega d\tau+\int_\Omega (wPe\theta)_{\tau=\tau_1} d\Omega$$

對時間間隔$\Delta\tau=\tau_2-\tau_1$，渦度及其比降變化與加權函數的變化相比較，非常微小時，只要把時間間隔取小值，時間積分只對加權函數即可，當極座標原點 P 在邊界上時，上式可改寫成

$$\gamma\theta(0,\tau_2)Pe+\int_\Gamma\int_{\tau_1}^{\tau_2}\theta w_{,n}\, d\tau d\Gamma=\int_\Gamma\int_{\tau_1}^{\tau_2}\theta_{,n}\, d\tau d\Gamma-\int_\Omega\int_{\tau_1}^{\tau_2}Peu_j\theta_{,j} w\, d\tau d\Omega+\int_\Omega (wPe\theta)_{\tau=\tau_1} d\Omega$$

在邊界上 γ=0.5，在領域內時 γ=1。

利用一定元素將上式差分，同時對時間採階梯狀變化，可得

$$\gamma\theta_i(\tau_2)Pe+\sum_{j=1}^{j=n}\int_\Gamma\theta\int_{\tau_1}^{\tau_2}w_{,n}\, d\tau d\Gamma=$$

$$\sum_{j=1}^{j=n}\int_{\Gamma_j}\theta_{,n}\int_{\tau_1}^{\tau_2}w\, d\tau d\Gamma-\sum_{j=1}^{j=m}\int_\Omega Peu_j\theta_{,j}\int_{\tau_1}^{\tau_2}w\, d\tau d\Omega+\sum_{j=1}^{j=m}\int_\Omega (wPe\theta)_{\tau=\tau_1} d\Omega$$

加權函數及其比降 $w_{,n}$ 的時間積分可以下式求得

$$\int_{\tau_1}^{\tau_2}w\, d\tau=\frac{Pe}{4\pi}\int_a^\infty\frac{1}{\eta}\exp(-\eta)\, d\eta=\frac{Pe}{4\pi}Ei(a)$$

$$\int_{\tau_1}^{\tau_2}w_{,n}\, d\tau d\Gamma=-\frac{Pe}{2\pi r}\frac{\partial r}{\partial n}\exp(-a)$$

$$Ei(a)=-\ln(a)-Ec-\sum\frac{(-a)^n}{n\cdot n!}$$

$$Ec=0.5772157\cdots\cdots$$

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